12 photons and phonons

12.1: The sun's temperature
Measured from the time when the first rays of sunshine appear above the horizon until the moment when the sun is fully visible, sunrise lasts 2.1 minutes. Based on this information, and assuming that the earth and the sun are "black bodies", can you estimate the temperature of the sun?

12.2: Radiation in a cavity
Exercise 9.9 from Reif's book

12.3: Detailed balance
Exercise 9.14 from Reif's book
Note: At 25 C, the vapor pressure of water is 3165 Pa.

12.4: Phonons on a string
Exercise 6.3 from Sethna's book

Suggested problems for further study:

12.5: Temperature of the earth
Exercise 9.13 from Reif's book

12.6: Photon fluctuations
Calculate the relative fluctuations $\Delta n_j/\bar{n}_j$ of the number of photons n_j for a photon mode at frequency ω_j.

12.7: Debye approximation
Use the Debye approximation for the normal mode spectrum of a solid to calculate

(a) the Helmholtz Free energy F;

(b) the entropy S;

Express your answers in terms of the Debye temperature Θ_D and the Debye function

$$D(y) = \frac{3}{y^3}\int_0^y \frac{x^3 dx}{e^x - 1}.$$

12.8: Expansion of a solid

The phonon analysis you have seen in lecture describes the contribution of the (quantized) motion of lattice atoms to the (free) energy of a solid. There is another contribution to the energy of a solid, which arises from the potential energy between the lattice atoms at their equilibrium positions. It depends on the total volume only, and, close to its minimum, it can be written

$$\mathcal{H}_0 = \frac{(V - V_0)^2}{2\kappa_0 V_0}.$$

(a) If you can neglect the effect of lattice vibrations, show that $\kappa_0 = -(1/V)(\partial V/\partial p)$ is the compressibility of the solid.

The combined effect of the potential energy $\mathcal{H}_0$ and the lattice vibrations can be studied through the Hamiltonian

$$\mathcal{H} = \frac{(V - V_0)^2}{2\kappa_0 V_0} + \sum_j \hbar\omega_j(n_j + 1/2),$$

where the index j labels the normal modes of the lattice and ω_j is the corresponding frequency.

(b) Show that the inclusion of the term $\mathcal{H}_0$ does not change the heat capacity C_V (*i.e.*,, show that a calculation of C_V which is based on the contribution from lattice vibrations only is still valid).

(c) Using the Debye approximation, show that the Helmholtz free energy F is given by

$$F = \frac{(V - V_0)^2}{2\kappa_0 V_0} + \frac{9}{8}Nk\Theta_D + NkT[3\ln(1 - e^{-\Theta_D/T}) - D(\Theta_D/T)],$$

where $D(y)$ is the Debye function, see Ex. 12.7.

(d) The pressure in the solid may be calculated from the thermodynamic relation $p = -(\partial F/\partial V)_T$. Here, one has to take into account that the Debye temperature Θ_D depends on volume. Assuming a linear dispersion relation for the normal modes of the lattice, the Debye temperature is given by

$$k\Theta_D = \hbar c(6\pi^2 N/V)^{1/3},$$

where c is the sound velocity in the solid (taken to be volume independent). Calculate the pressure p.

(e) Calculate the isothermal compressibility $\kappa = -(1/V)(\partial V/\partial p)_T$. You may express your answer in terms of V, T, and Θ_D.

(f) Show that, for this solid, the thermal expansion coefficient $\beta = (1/V)(\partial V/\partial T)_p$, the isothermal compressibility κ, and the heat capacity C_V are related as

$$\beta = \kappa \left(\frac{\partial p}{\partial T}\right)_V = \frac{\kappa C_V}{3V}.$$

12.9: Photon Entropy

The mean energy $\bar{E}$ and the pressure p for a photon gas in a cavity of volume V and at temperature T are

$$\begin{aligned} \bar{E} &= \frac{\pi^2}{15} V \frac{(kT)^4}{(c\hbar)^3}, \\ p &= \frac{\pi^2}{45} \frac{(kT)^4}{(c\hbar)^3}. \end{aligned}$$

(a) Using the thermodynamic relation $TdS = d\bar{E} + pdV$, derive an expression for the entropy S of the photon gas. Express S as a function of T and V.

(b) If the volume V of the cavity is increased adiabatically by a factor eight, what is the change of the temperature of the radiation in the cavity?

12.10: Phonon modes

Consider a one-dimensional lattice of N atoms of mass m. We consider "periodic boundary conditions", i.e., the atoms form a ring so that the Nth atom is connected to the 1st one. The equilibrium distance between the atoms is a. Denoting the position and momentum of the jth atom by x_j and p_j, respectively, the Hamiltonian for the atoms is

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2m} + \frac{k}{2} \sum_{j=1}^{N-1} (x_{j+1} - x_j - a)^2 + \frac{k}{2} (x_1 - x_N + (N-1)a)^2.$$

(Hence, the atoms can be seen as a collection of masses m connected by springs with spring constant k.)

By Fourier transform, we change variables to "normal mode coordinates"

$$q_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{ikja}(x_j - aj), \quad p_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-ikja} p_j.$$

Because of the periodic boundary conditions, the allowed wavenumbers are $k = 2n\pi/L$, with integer. Note, however, that the wavenumbers k and $k + 2\pi/a$ describe the same normal mode, so that, effectively, we only have N different wavenumbers. The inverse relations are

$$x_j = aj + \frac{1}{\sqrt{N}} \sum_{k} e^{-ikja} q_k, \quad p_j = \frac{1}{\sqrt{N}} \sum_{k} e^{ikja} p_k,$$

where the summation is over one complete set of wavenumbers, e.g., $k = 2\pi n/L$ with $n = 0, 1, \ldots, N-1$.

(a) Show that the normal-mode coordinates obey the usual quantum-mechanical commutation relations.

(b) Show that $q_k = q_{-k}^*$. Which normal modes have real amplitudes, and which ones do have complex amplitudes? (Distinguish between the cases of N even and N odd.) Are all modes independent? What is the total number of degrees of freedom in the normal-mode representation?

(c) Write H in terms of the normal-mode coordinates. Keep independent normal-mode coordinates only.

(d) What is the frequency ω_k of a normal mode with wavenumber k?

(e) What is the density $\sigma(\omega)$ of lattice vibration modes?