4 Statistical Thermodynamics (2)

4.1: Shannon entropy

Exercise 4.4 from Sethna's book

4.2: Black hole thermodynamics

Exercise 4.8 from Sethna's book;
Black body radiation will be discussed later in this course. For this exercise you only need to use the fact that, according to the radiation it emits, a black hole is in equilibrium at temperature T_{bh}.

4.3: Equilibrium fluctuations

Consider a monatomic ideal gas consisting of N gas atoms in a container of volume V.

(a) How does the number $\Omega(E, V, N)$ of accessible microstates at energy E and volume V depend on E and on V. You may ignore any numerical prefactors that depend on N only.

(b) The gas is brought in thermal equilibrium with a "heat reservoir" at temperature T. (A heat reservoir is a macroscopic system that is so large that its temperature remains unaffected if energy is absorbed or emitted.) As a result, the energy E of the gas shows random fluctuations as a function of time. For these fluctuations, find the average energy $\bar{E}$ and the variance of the energy $\operatorname{var} E = \overline{(E - \bar{E})^2}$.

(c) The gas is brought in thermal equilibrium with a heat and pressure bath at temperature T and pressure p. (A "pressure reservoir" is a macroscopic system that is so large that its pressure remains unaffected if its volume is changed. Our local atmosphere is an example of a heat and pressure reservoir.) As a result, the energy E and the volume V of the gas show random fluctuations as a function of time. For these fluctuations, find the averages $\bar{E}$ and $\bar{V}$ and the variances $\operatorname{var} E$ and $\operatorname{var} V$. Are the fluctuations of energy and volume statistically independent?

4.4: Electrons in a metal

At low temperatures, the heat capacity of a metal is of the form

$$C = aT,$$

where a is a proportionality constant. Using this information, find the increase of entropy if the temperature of the metal is raised from T_1 to T_2.

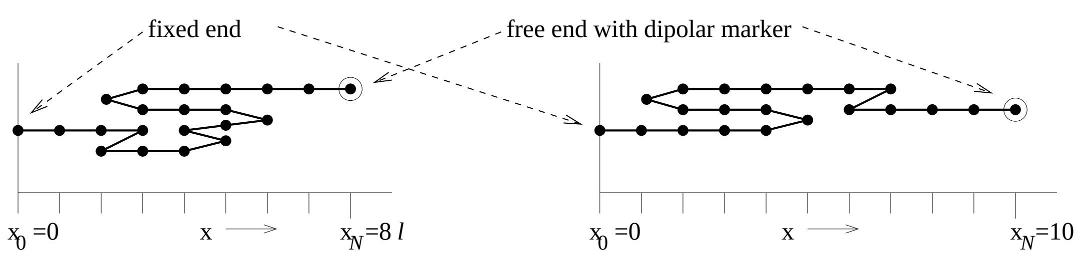
Suggested problems for further study:

4.5: Pulling on a polymer

With the help of (bio)chemical methods, it has become possible to study the mechanical properties of long molecules or polymers, such as proteins or DNA. In this exercise, you are asked to consider one such molecule. Although a polymer is only one molecule, it has many degrees of freedom because it is flexible. That's why we can use methods from statistical mechanics to describe polymers.

One end of the polymer is attached to a substrate. To the other end, which is free to move, a dipolar marker is attached. With the help of a uniform electric-field gradient, a constant force can be exerted on the dipole.

We will model the polymer as a one-dimensional chain of N links of length l. Each link can point to the left or to the right. Two examples of possible configurations of the polymer are shown in the figure below for $N = 20$. The real polymer, of course, has a much larger number of links N. (In the figure, for clarity a second dimension has been added.) One end of the chain is kept at fixed position $x_0 = 0$ (since it is attached to the substrate), whereas the other end, at position x_N, is free to move.



The energy of the polymer is

$$E = -a x_N.$$

The constant a is positive and proportional to the marker's dipole moment and the electric field gradient. Since $-Nl \leq x_N \leq Nl$, the energy E can take values between $-Nal$ and Nal.

(a) Show that for $N \gg 1$ the number of "microstates" (configurations) $\Omega(E)$ for which the energy of the polymer is between E and $E + \delta E$ is equal to

$$\Omega(E) = \frac{2^N \delta E}{al\sqrt{2\pi N}} e^{-E^2/2Nl^2a^2},$$

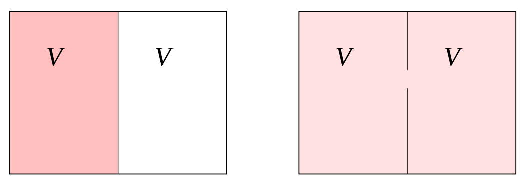
if $al \ll \delta E \ll E$ and $|E| \ll Nal$.

(b) Find the entropy S of the polymer at energy E.

(c) The polymer is immersed in a solution at temperature T. If the polymer and the surrounding liquid are in thermal equilibrium, what is the (average) energy $\bar{E}$ of the polymer? What are the fluctuations of the energy?

(d) What is the force the polymer exerts on the marker? Formulate your answer in terms of the polymer's extension x_N, the temperature T, the link length l, and the number of links N.

4.6: Ideal gas



(a) Two thermally isolated containers of volume V each are separated by a wall, see figure. One container contains N molecules of an ideal gas in equilibrium, whereas the other container is empty. A small hole is made in the wall separating the containers and one waits until a new equilibrium has been reached. What is the increase ΔS of the entropy S for this process? Is the process of puncturing the wall reversible or irreversible?

(b) Consider the case that, initially, each containers is filled with N molecules of the same ideal gas. Again the wall is punctured. What is the change in entropy? In this case, is the process of puncturing the wall reversible or irreversible?

(c) Now consider the case that, initially, the containers are filled with N molecules of different ideal gases. (That is, each container contains N gas molecules.) Again the wall is punctured. What is the change in entropy? Is the process of puncturing the wall reversible or irreversible?

4.7: Absorption of heat from a heat reservoir

Exercise 3.4 from Reif's book. Note: a "heat reservoir" is a macroscopic system that is so large that its temperature remains unaffected if energy is absorbed or emitted.

4.8: Mixture of gases

Exercise 3.5 from Reif's book. Add:

(c) If both gases consist of monatomic molecules with masses m_1 and m_2, respectively (e.g., He and Ar), find an explicit expression for the entropy S.

(d) What are the heat capacities C_p and C_V at constant pressure and constant volume, respectively?

4.9: Heat capacity of a ferromagnet

Exercise 4.4 from Reif's book.